MOND-LIKE GRAVITY BY GIVING MASS TO CURVED SPACETIME (NOTES, V1.1)

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ABSTRACT

This document contains some notes on a proposed mechanism for MoND-like gravity. Rather than using a non-local Lagrangian, we assign spacetime a small mass that depends on the local curvature (or in Newtonian terms, the tidal acceleration). This naturally gives a gravitational acceleration that increases as $1/r$ for very large r, similar to MoND. However, for smaller masses this model predicts that Newtonian behavior would persist at much larger length-scales than MoND does.

Index Terms— Gravity, MoND

1. INTRODUCTION

MoND– a form of gravity that decreases as $1/r$ once the gravitational acceleration is below a critical threshold– has been proposed as a possible alternative to dark matter but has not been considered mainstream because it has been hard to see how it could arise from physics. We will show that it is possible to obtain a form of gravity with quite similar characteristics to MoND, if we assign a mass to spacetime that is curved. In our scheme the boundary between normal and MoND-like gravity depends on the value of the tidal accleration in units of acceleration per meter, or s^{-2}, which is locally measurable, rather than acceleration in ms^{-2} (which is not).

2. TIDAL ACCLERATION

We need a definition of tidal accleration that is covariant so that the GR extension can be done. I believe tidal accleration is known to be covariant. There may be a number of ways to reduce tidal accleration to a scalar, and in these notes we won't distinguish between them. For now, just let the tidal accleration, or curvature, at a moment in spacetime be some quantity u such that, at radius r from an isolated mass m,

$$u = Gm/r^3 \tag{1}$$

with G the gravitational constant. The units of u are acceleration divided by distance, or s^{-2}.

As far as the rest of this document is concerned, we just need to know that u has the property above.

3. EFFECTIVE MASS

An isolated mass m will have a halo of extra vacuum density around it. At a distance r from the mass, the effective mass will be m plus the extra mass from the vacuum inside a sphere of radius r. Remember that inside a hollow sphere we feel no gravitational accleration, but outside the sphere it behaves as if the mass is located at the center. Let the effective mass at radius r be

$$\text{effective mass at r} = mf(x(r)) \tag{2}$$

where $x(r)$ is a scaled version of the radius, defined as

$$x = \alpha m^{-1/3} r, \tag{3}$$

and f is some function satisfying $f(0) = 1$ that becomes similar to $f(x) = x$ for large x, giving a MoND-like property. The sole adjustable parameter of our model is α. We will also have some choices about the functional form of f, subject to some constraints that will become clear later. The choice to insert a factor of $m^{-1/3}$ is dictated by the need for local measurability.

4. TIDAL ACCLERATION TO DENSITY

Let the tidal acceleration u, in units of s^{-2}, be as discussed in Section 2. Let us define a rescaled tidal acceleration

$$C = \frac{u}{G\alpha^3} \tag{4}$$

where G is the gravitational consant; this will keep our equations simple. (We use the letter C for curvature; we used u for for the unscaled tidal acceleration to avoid confusion with the speed of light c). At a radius r from mass m, if we were to ignore the extra vacuum density inside r we would have

$$C = \frac{u}{G\alpha^3} = \frac{m}{r^3\alpha^3} = \frac{1}{x^3}. \tag{5}$$

but this needs to be corrected for the vacuum density inside the sphere, which makes the central mass appear heavier by a factor of $f(x)$, so:

$$C = \frac{f(x)}{x^3}. \tag{6}$$

We introduce a scalar function $H(\cdot)$ that gives the density of the vacuum as a function of the tidal acceleration. The relationship will be:

$$\text{density of vacuum} = \frac{\alpha^3}{4\pi} H(C), \tag{7}$$

where the constant factor $\alpha^3/4\pi$ was introduced for mathematical convenience but could easily be absorbed into H. The idea is that H would reflect some physical process that gives curved vacuum a density, perhaps related to the paths taken by 'phantom particles' arising from quantum mechanics.

5. SETTING UP EQUATION FOR H

We can solve for $H(\cdot)$ by considering an isolated mass m, and giving the vacuum the density that's required to make the effective mass at a particular radius vary as $mf(x)$. (x varies with r).

We need to work out the mass of the vacuum in a hollow sphere at radius r and thickness dr. With $x = \alpha m^{-1/3} r$, the density of the vacuum is:

$$\text{density} = \frac{\alpha^3}{4\pi} H\left(\frac{f(x)}{x^3}\right) \tag{8}$$

(in kg/m^3). The volume of the hollow sphere is $4\pi r^2 dr$, so:

$$\text{mass of hollow sphere} = \alpha^3 r^2 H\left(\frac{f(x)}{x^3}\right) dr. \tag{9}$$

The amount by which the effective mass of our object is needs to vary from r to $r + dr$ is

$$\frac{d}{dr} mf(x) = mf'(x)\frac{dx}{dr} \tag{10}$$

Setting this to equal the mass of the hollow sphere computed above, we have:

$$mf'(x)\frac{dx}{dr} = \alpha^3 r^2 H\left(\frac{f(x)}{x^3}\right) \tag{11}$$

$$mf'(x)\alpha m^{-1/3} = \alpha^3 r^2 H\left(\frac{f(x)}{x^3}\right) \tag{12}$$

$$f'(x) = x^2 H\left(\frac{f(x)}{x^3}\right). \tag{13}$$

6. CONDITIONS FOR H TO EXIST

Before trying to solve Equation (13) we would like to point out some trivial solutions and limits. The solution that corresponds to Newtonian gravity is $f(x) = 1$, $H(C) = 0$. If we want $f(x) \to x$ as $x \to \infty$ (as in MoND), then (13) becomes $1 = x^2 H\left(\frac{x}{x^3}\right)$ for large x, which requires $H(C) \to C$ for small C.

A fairly general condintion for (13) to be solvable, is for the function

$$g(x) = \frac{f(x)}{x^3} \tag{14}$$

to be invertible, i.e. for the function $g^{-1}(C)$ to exist: in that case we can let

$$H(C) = \frac{f'(g^{-1}(C))}{g^{-1}(C)^2}. \tag{15}$$

Since $f(x) \to x$ for large x, $g(x)$ approaches $\frac{1}{x^2}$ for large x, implying $g'(x) < 0$. We can ensure $g(\cdot)$ is invertible by requiring $g'(x) < 0$ for all $x > 0$. So

$$g'(x) = \frac{f'(x)}{x^3} - 3\frac{f(x)}{x^4} < 0 \tag{16}$$

for $x > 0$. Multiplying by x^4 and rearranging,

$$xf'(x) < 3f(x). \tag{17}$$

All the functions $f(\cdot)$ that we are considering satisfy $f(x) \geq x$ and $f'(x) \leq 1$, satisfying (17), so invertibility of g is not a problem. We can, however, rule out equations where $f(x)$ rises "too fast" near $x = 0$, for instance $f(x) = \sqrt{(x)} + x$.

7. DISCUSSION ON CHOICE OF F

It seems to be difficult to find easy functional forms for both f and H simultaneously. We considered various forms for f, including $f(x) = 1+x$, $f(x) = 1+\sqrt{1+x^2}$, and others. We are inclined towards $f(x) = 1+x$ for its simplicity, because it leads to a relatively simple relationship between density and curvature, and because it lets the density increase as fast as possible (in terms of powers of x) for small x. The reason we feel this is attractive is because it increases the chance that this mechanism could get rid of the singularity inside black holes. All our analysis is in the Newtonian regime and in this case it's not possible for a vanishingly small central mass to have a large effective mass in a finite radius, but the hope is that this might change due to relativistic effects.

8. SOLUTIONS FOR H

We'll consider a solution for H in the case where $f(x) = 1 + x$. Equation (13) becomes:

$$1 = x^2 H\left(\frac{1+x}{x^3}\right). \tag{18}$$

Remember that H returns a density; this density needs to equal $1/x^2$ to satisfy (18). We can rewrite this as:

$$H\left(D + D^{3/2}\right) = D, \tag{19}$$

where $D = \frac{1}{x^2}$ is the (scaled) density. So H is the inverse function of $g(D) = D + D^{3/2}$. This function is clearly invertible for $D \geq 0$ because it is monotonically increasing. Unfortunately it does not have a particuarly nice functional form; the functional form arises from the formulas for solving cubic equations, which involves a difference of imaginary numbers (canceling to leave a real result). The equation does not look particularly enlightening so we won't write it down here.

9. COSMOLOGICAL IMPLICATIONS

We don't have the time or the right background to do any detailed checks whether this model correctly predicts observations, but we would like to point out that on smaller scales, this model predicts closer-to-Newtonian behavior than MoND does.

It has been estimated that the total (DM + baryonic) of the galaxy within 129,000 light years from the center is $1.5 * 10^{12} M_\odot$ ($M_\odot$ is the mass of the Sun), while the baryonic mass of the galaxy is about $1.7 * 10^{11} M_\odot$.

Attributing the excess to our model, that would mean $f(x) = 1 + x \simeq 8.8$ at 129,000 ly. The distance at which $f(x) = 2$, i.e. the dark and baryonic masses are the same (assuming all the baryonic mass is located in the center) would be 129,000 ly / 7.8 $\simeq$ 16,000 ly. This length-scale varies as $m^{1/3}$ with the mass of the central object, so the corresponding length-scale for our Sun would be $16,000ly \times (1.7 \times 10^{11})^{-1/3} = 2.8ly \simeq 1pc$. That is: the gravitational acceleration at 1 parsec from the Sun would be double the Newtonian prediction. This is different from MoND which predicts non-Newtonian gravity at a much smaller distance of about 0.03pc.

Applying this same reasoning to a single molecule of hydrogen, the "critical distance" at which $f(x) = 2$ would equal $1pc * \sqrt[3]{M_{H_2}/M_\odot}$, or $1pc * \sqrt[3]{3 \times 10^{-27}/2 \times 10^{30}}$, which comes to 3.5mm. In a cloud of hydrogen gas the tidal forces from different hydrogen atoms will tend to cancel out at large scales, but we can apply our model out to around the distance by which hydrogen atoms are separated, e.g. at around 1 atom per cm^3 (the approximate density of H_2 gas in our galaxy) we might get around a factor of 3 in mass from this mechanism.

In fact, it's not hard to show that the effective mass per unit volume of evenly distributed objects depends on the average density but not on the size of the objects; this should be true from atomic to galactic scales. If $d_{\rm tot}$ is the total baryonic plus gravitational density and d_b is the baryonic density, a system will tend to obey the relationship $d_{\rm tot} = d_b(1 + d_b^{-1/3})$ (given appropriate units).